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On the triangular Potts model with two- and three-site interactions

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Abstract. The equivalence of the triangular Potts model having two- and three-site interactions with a 20-vertex Kelland model is rederived using a graphical method. The conjectured critical point of this Potts model is shown to agree with the known results in two instances.

1. Introduction

The Potts model (Potts 1952) has remained to this date one of the most intriguing lattice statistical models of phase transitions. While its exact solution is not yet known, significant progress has been made in recent years in exact analyses of its properties. The breakthrough came in 1971 when Temperley and Lieb (1971) established a remarkable equivalence of the nearest-neighbour Potts model on the square lattice with an ice-rule model, a fact that made possible the exact determination of its critical properties (Baxter 1973). These considerations have recently been extended to the Potts model with two- and three-site interactions (Baxter *et al* 1978). In these analyses an operator method has been used to establish the equivalence of the Potts model with an ice-rule model. A simpler and more direct graphical analysis for proving this equivalence was later developed by Baxter *et al* (1976) for the pure two-site problem. In view of the usefulness and richness of the new results of Baxter *et al* (1978), it appears desirable to extend the graphical approach to models with two- and three-site interactions. This is the subject matter of the present paper.

We shall proceed in a way which differs slightly from that of Baxter *et al* (1976). We define, in § 2, a five-vertex model on the triangular lattice, and show in § 3 that this vertex model is equivalent to the Potts model under consideration. A simple symmetry of the vertex model then leads to a duality relation for the Potts model, which, in turn, determines the Potts critical point. This conjectured critical point is shown to reduce to the known exact results in two instances. In § 4 we show that the five-vertex model is also equivalent to a Kelland (1974) model. It follows that the Potts model with two- and three-site interactions is equivalent to an ice-rule Kelland model, thus rederiving the result obtained by Baxter *et al* (1978).

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2. Five-vertex model on the triangular lattice

Consider a triangular lattice $\mathscr{L}'$ of N sites. Cover all edges of $\mathscr{L}'$ with bonds and join the ends of bonds so that the bonds form non-crossing paths. A typical joining of the bonds is shown in figure 1. Note that the bonds form closed, non-intersecting polygons. The six bonds incident at a vertex can join in only five distinct ways. These five configurations are shown in figure 2.

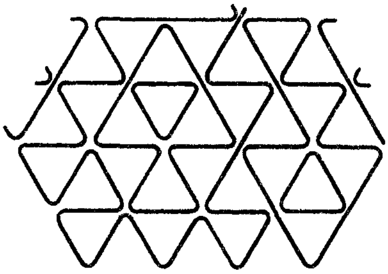

Figure 1. A typical bond graph on $\mathscr{L}'$. The bonds form closed, non-intersecting polygons.

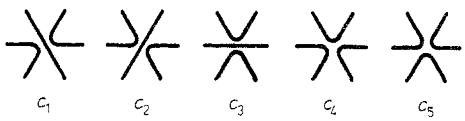

Figure 2. Vertex configurations of the five-vertex model.

Associate weights c_i, $i = 1, 2, \ldots, 5$, with these five configurations as shown in figure 2. Further, with each polygon on $\mathscr{L}'$ we associate a weight z. The partition generating function for this five-vertex model is defined to be

$$Z_{12345} \equiv Z(z; c_1, c_2, c_3, c_4, c_5)$$
$$= \sum_P z^p \prod_{i=1}^{5} c_i^{n_i} \tag{1}$$

where the summation is taken over the 5^N polygonal configurations, or bond joinings, on $\mathscr{L}'$, n_i is the number of vertices of type i satisfying

$$n_1 + n_2 + n_3 + n_4 + n_5 = N \tag{2}$$

and p is the number of polygons. The partition function (1) possesses the obvious 60° rotational symmetry

$$Z_{12345} = Z_{31254} = Z_{23145} = Z_{12354} = Z_{31245} = Z_{23154}. \tag{3}$$

It follows that Z is invariant under the cyclic permutations of the indices 1, 2, 3, and/or 4, 5.

3. Reduction to a Potts model

We now show that the five-vertex model (1) is equivalent to the Potts model with two- and three-site interactions considered by Baxter *et al* (1978).

There are two kinds of faces in the triangular lattice $\mathscr{L}'$, namely, the up-pointing and down-pointing triangles. Following Baxter *et al* (1976), we shade one kind of the faces, say the down-pointing triangles, and regard such shaded areas as 'land', and the remaining unshaded areas as 'water'. Then, as shown in figure 3, a typical polygonal configuration P will consist of *connected* lands surrounded by water.

Next, we place a site at the middle of each of the N shaded triangles, and join as shown in figure 3 the two or three neighbouring sites whose lands are connected. The connecting lines are either boomerang- or Y-shaped. Consider now the triangular lattice $\mathscr{L}$ formed by these N sites. The partition function (1) can also be interpreted as defined on $\mathscr{L}$ as follows.

(1) Each of the N up-pointing triangular faces of $\mathscr{L}$ can independently take one of the five configurations shown in figure 4. This specifies the configuration P.

(2) The numbers c_i and n_i are, respectively, the weight and multiplicity of the ith vertex configuration, $i = 1, 2, \ldots, 5$, in P.

(3) $p = C + S$, where C and S are, respectively, the numbers of connected components, including isolated sites, and circuits in P.

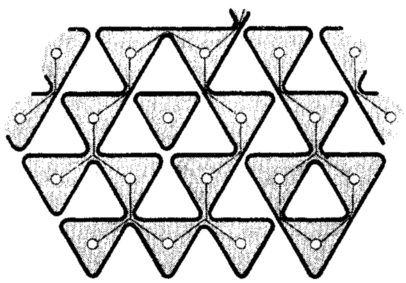

Figure 3. The same bond graph as in figure 1. The down-pointing triangular faces are shaded showing connected lands surrounded by water. The circles form a triangular lattice $\mathscr{L}$.

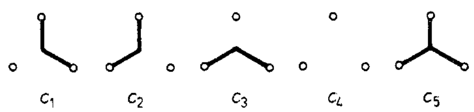

Figure 4. The five possible bond configurations for the up-pointing triangular faces of $\mathscr{L}$.

Here, use has been made of the fact that, for a given P, each closed polygon on $\mathscr{L}'$ is the outside perimeter of either a circuit or a connected component of the associated configuration on $\mathscr{L}$.

Consider a q-state Potts model on $\mathscr{L}$ whose interactions consist of two-site interactions ϵ_1, ϵ_2, ϵ_3 and three-site interactions ϵ among every three sites surrounding an up-pointing (triangular) face. This is shown in figure 5. The Hamiltonian now reads

$$\mathscr{H} = \sum_{\Delta} E_{abc} \tag{4}$$

where the summation is over all up-pointing faces of $\mathscr{L}$ and

$$E_{abc} = -(\epsilon_1 \delta_{bc} + \epsilon_2 \delta_{ca} + \epsilon_3 \delta_{ab} + \epsilon \delta_{abc}). \tag{5}$$

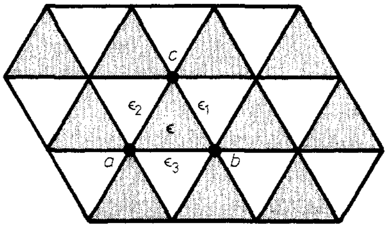

Figure 5. The Potts model on $\mathscr{L}$ with two- and three-site interactions at each shaded triangle.

Here, $\delta_{ab} = \delta_{\mathrm{Kr}}(\xi_a, \xi_b)$, $\delta_{abc} = \delta_{ab}\delta_{bc}$, and $\xi_a = 1, 2, \ldots, q$ refer to the spin state at the site a.

Following Baxter *et al* (1978), we write

$$\exp(-\beta E_{abc}) = 1 + f_1\delta_{bc} + f_2\delta_{ca} + f_3\delta_{ab} + y\delta_{abc} \tag{6}$$

where

$$\begin{aligned} f_i &= \exp(\beta\epsilon_i) - 1 \\ g &= \exp(\beta\epsilon) - 1 \\ y &= f_1f_2 + f_2f_3 + f_3f_1 + f_1f_2f_3 + g(1+f_1)(1+f_2)(1+f_3) \end{aligned} \tag{7}$$

and $\beta = 1/kT$. The partition function of the Potts model is

$$Z_{\text{Potts}}(q; f_1, f_2, f_3, y) = \sum \prod_{\Delta} (1 + f_1\delta_{bc} + f_2\delta_{ca} + f_3\delta_{ab} + y\delta_{abc}) \tag{8}$$

where the summation is over the q^N spin states. The product is taken over the N shaded triangles shown in figure 5.

Expand the product in (8). A natural graphical representation of the expansion is as follows. To each factor $f_i\delta_{ab}$ associate a boomerang-shaped bond connecting the sites a and b, and to each factor $y\delta_{abc}$ associate a Y-shaped bond connecting the sites a, b and c. Since these are precisely the configurations shown in figure 4, we can write, as in (1),

$$Z_{\text{Potts}} = \sum_P q^C f_1^{n_1} f_2^{n_2} f_3^{n_3} y^{n_5} \tag{9}$$

where the summation is taken over the 5^N configurations P on $\mathscr{L}$. Also, since f_i connects two sites and each Y connects three sites, we have the Euler relation

$$N + S = C + n_1 + n_2 + n_3 + 2n_5. \tag{10}$$

Eliminating N and S from (2), (10) and the relation $p = C + S$, we obtain

$$C = \tfrac{1}{2}(p + n_4 - n_5). \tag{11}$$

Substituting (11) into (9) and comparing with (1), we arrive at the identity

$$Z_{\text{Potts}}(q; f_1, f_2, f_3, y) = Z(\sqrt{q}; f_1, f_2, f_3, \sqrt{q}, y/\sqrt{q}). \tag{12}$$

This states that the Potts model (4) is equivalent to the five-vertex model (1), a result we set out to prove. Note that there is no loss of generality in taking $z = c_4$ in (1), since Z is homogeneous in c_i.

The invariance of Z_{12345} under the interchange of the indices 4 and 5 now implies the following duality relation for Z_{Potts} (Baxter *et al* 1978):

$$Z_{\text{Potts}}(q; f_1, f_2, f_3, y) = (y/q)^N Z_{\text{Potts}}(q; f_1', f_2', f_3', y'), \tag{13}$$

where

$$f_i' = qf_i/y \qquad y' = q^2/y. \tag{14}$$

The transformation (13) maps the partition function at a temperature $T > T_c$ into one at another temperature $T < T_c$, and vice versa, where T_c is determined by the fixed point

$$y = q \tag{15}$$

of the transformation. In the isotropic case ($\epsilon_1 = \epsilon_2 = \epsilon_3$) (15) reads

$$3f^2 + f^3 + g(1+f)^3 = q \tag{16}$$

and we plot (16) in figure 6 to give T_c as a function of $\alpha \equiv \epsilon/\epsilon_1$. Along the $\alpha = 0$ axis the T_c in figure 6 is known to be exact and unique (Hintermann *et al* 1978). If, for $\alpha \neq 0$, one assumes the transition also to be unique, then the critical point is given by (16). We expect a similar uniqueness argument to lead to the critical condition (15) in the general anisotropic case. Indeed, the general Potts model (4) is exactly soluble for $q = 2$. In this case the state ξ_a may be described by the Ising variables $\sigma_a = \pm 1$ and we write $\delta_{ab} = \frac{1}{2}(1 + \sigma_a \sigma_b)$. The Potts model is then exactly equivalent to a triangular Ising model whose interactions are $J_i = \frac{1}{2}\epsilon_i + \frac{1}{4}\epsilon$. From the known solution of the triangular Ising model (Houtappel 1950), one verifies that its critical condition is indeed (15) in the region $\epsilon + \epsilon_i + \epsilon_j \geq 0$, $i \neq j$, or

$$(1+g)(1+f_i)(1+f_j) \geq 1 \qquad i \neq j. \tag{17}$$

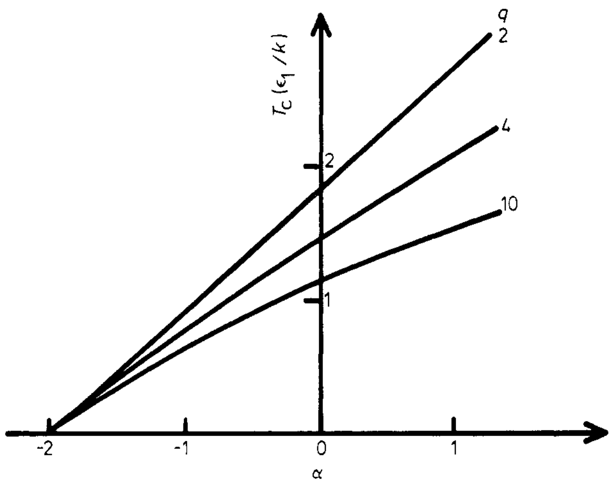

Figure 6. The transition temperature T_c in the isotropic case, T_c in units of ϵ_1/k and $\alpha = \epsilon/\epsilon_1$. The straight line $\alpha + 2 = T_c \ln 3$ for $q = 2$ is exact.

Since the Ising critical condition is different from (15) outside the region (17), the validity of (15) will generally be limited. It appears safe, however, to expect (15) to hold at least for positive g and f_i. We note that (15) indeed reduces to the exact results of Hintermann *et al* (1978) for $g=0$ and $f_i \geqslant 0$.

4. Equivalence with an ice-rule model

In this section we show that the five-vertex model (1) is also equivalent to an ice-rule model, thereby deriving the equivalence of the latter with the Potts model.

Consider the partition generating function (1) for the five-vertex model. Write

$$z = t^3 + t^{-3} \tag{18}$$

and expand the factor $(t^3+t^{-3})^p$ in (1). Following Baxter *et al* (1976), a natural graphical representation of this expansion is to direct the polygons in P and associate the weights t^3 and t^{-3} to the directed polygons. As shown in figure 7, let t^3 (t^{-3}) be the weight of a clockwisely (counterclockwisely) directed polygon. The polygonal weights $t^{\pm 3}$ can also be associated with the vertices with the following rule (Baxter *et al* 1976): each directed line turning an angle θ to the right (left) carries a weight $t^{3\theta/2\pi}(t^{-3\theta/2\pi})$. This leads us to consider a vertex problem on $\mathscr{L}'$ whose edges are directed. Since there are always three arrows out and three arrows in at each vertex, we are led to the Kelland (1974) model, namely, the 20-vertex ice-rule model on $\mathscr{L}'$. Collecting the weights of those vertices having the same arrow arrangement, we obtain, as shown in figure 8, the following equivalence:

$$Z_{12345} = Z_{\text{Kelland}}(u_i, u_i'). \tag{19}$$

Here Z_{Kelland} is the partition function of the Kelland model. The vertex weights of the Kelland model are obtained from figure 8:

$$\begin{aligned}
&u_1 = c_2 \qquad u_2 = c_1 \qquad u_3 = c_3 \\
&u_4 = c_1 + c_2 + c_3 + c_4 t^3 + c_5 t^{-3} \qquad u_5 = c_3 t^2 + c_5 t^{-1} \\
&u_6 = c_3 t^{-2} + c_4 t \qquad u_7 = c_2 t^2 + c_4 t^{-1} \\
&u_8 = c_2 t^{-2} + c_5 t \qquad u_9 = c_1 t^2 + c_5 t^{-1} \\
&u_{10} = c_1 t^{-2} + c_4 t \qquad u_i' = u_i(t \to t^{-1}).
\end{aligned} \tag{20}$$

The configuration of u_i' is the same as that of u_i with all arrows reversed.

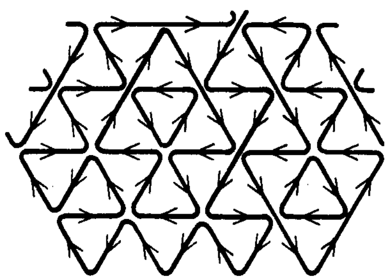

Figure 7. A typical directed polygonal configuration on $\mathscr{L}'$. Each polygon can be directed either clockwisely or counterclockwisely carrying respective weights t^3 and t^{-3}.

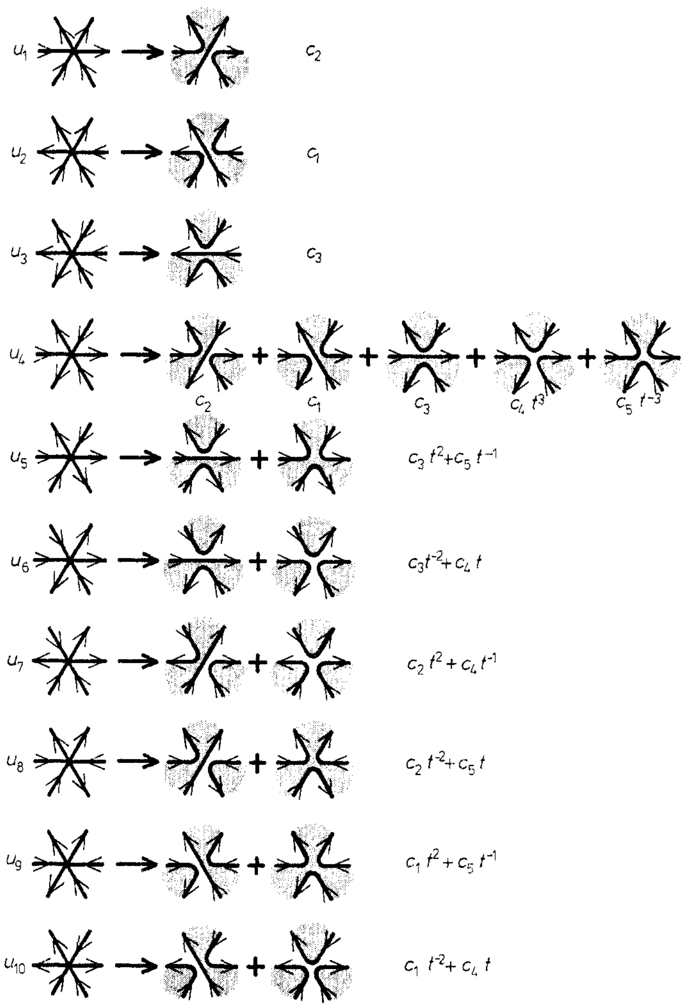

Figure 8. Vertex weights and vertex decompositions of a Kelland model.

The equivalence (20) is valid for general c_i and t, or z. Specialising to the Potts model for which, from (12),

$$z = \sqrt{q} \qquad c_1 = f_1 \qquad c_2 = f_2 \qquad c_3 = f_3 \qquad c_4 = \sqrt{q} \qquad c_5 = y/\sqrt{q}, \tag{21}$$

(19) leads to an equivalence of the Potts model with a Kelland model. If, without changing Z_{Kelland}, we further introduce in (20) a factor $t^{1/2}$ ($t^{-1/2}$) to each arrow entering (leaving) a vertex in the three-direction or leaving (entering) in the one-direction, the resulting u_i and u_i' reduce exactly to those obtained by Baxter *et al* (1978). We have thus rederived their result.

5. Summary

We have established from a graphical consideration the equivalence of the triangular Potts model (4) with a Kelland model whose parameters u_i and u_i' are given by (20) and

(21). The conjectured critical point of this Potts model,

$$f_1f_2+f_2f_3+f_3f_1+f_1f_2f_3+g(1+f_1)(1+f_2)(1+f_3)=q, \tag{22}$$

agrees with the exact results for $q=2$ and/or for $g=0$.

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